

Note on q -extensions of Euler numbers and polynomials of higher order

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Abstract In [14] Ozden-Simsek-Cangul constructed generating functions of higher-order twisted (h, q) -extension of Euler polynomials and numbers, by using p -adic q -deformed fermionic integral on \mathbb{Z}_p . By applying their generating functions, they derived the complete sums of products of the twisted (h, q) -extension of Euler polynomials and numbers, see[13, 14]. In this paper we consider the new q -extension of Euler numbers and polynomials to be different which is treated by Ozden-Simsek-Cangul. From our q -Euler numbers and polynomials we derive some interesting identities and we construct q -Euler zeta functions which interpolate the new q -Euler numbers and polynomials at a negative integer. Furthermore we study Barnes' type q -Euler zeta functions. Finally we will derive the new formula for "sums products of q -Euler numbers and polynomials" by using fermionic p -adic q -integral on \mathbb{Z}_p .

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1. Introduction and notations

Throughout this paper we use the following notations. By \mathbb{Z}_p we denote the ring of p -adic rational integers, \mathbb{Q} denotes the field of rational numbers, \mathbb{Q}_p

denotes the field of p -adic rational numbers, \mathbb{C} denotes the complex number field, and \mathbb{C}_p denotes the completion of algebraic closure of \mathbb{Q}_p . Let ν_p be the normalized exponential valuation of \mathbb{C}_p with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q -extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p -adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q - 1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. In this paper we use the following notation:

$$[x]_q = [x : q] = \frac{1 - q^x}{1 - q}, \text{ cf. [1, 2, 3, 4, 5, 21] .}$$

Hence, $\lim_{q \rightarrow 1} [x] = x$ for any x with $|x|_p \leq 1$ in the present p -adic case. Let d be a fixed integer and let p be a fixed prime number. For any positive integer N , we set

$$\begin{aligned} X &= \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}), \\ X^* &= \bigcup_{\substack{0 < a < dp \\ (a,p)=1}} (a + dp\mathbb{Z}_p), \\ a + dp^N\mathbb{Z}_p &= \{x \in X \mid x \equiv a \pmod{dp^N}\}, \end{aligned}$$

where $a \in \mathbb{Z}$ lies in $0 \leq a < dp^N$. For any positive integer N ,

$$\mu_q(a + dp^N\mathbb{Z}_p) = \frac{q^a}{[dp^N]_q}$$

is known to be a distribution on X , cf.[1-19]. From this distribution we derive the p -adic q -integral on \mathbb{Z}_p as follows:

$$\int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} q^x f(x), \text{ where } f \in UD(\mathbb{Z}_p), \text{ see [1-22] .}$$

Higher-order twisted Bernoulli and Euler numbers and polynomials are studied by many authors (see for detail [1-20]). In [14] Ozden-Simsek-Cangul constructed generating functions of higher-order twisted (h, q) -extension of Euler polynomials and numbers, by using p -adic q -deformed fermionic integral on \mathbb{Z}_p . By applying their generating functions, they derived the complete sums of products of the twisted (h, q) -extension of Euler polynomials and numbers, see[13, 14]. In this paper we consider the new q -extension of Euler numbers and polynomials to be different which is treated by Ozden-Simsek-Cangul. From our q -Euler numbers and polynomials we derive some interesting identities and we construct q -Euler zeta functions which interpolate the new q -Euler numbers and polynomials at a negative integer. Furthermore we study Barnes' type q -Euler zeta functions. Finally we will derive the new formula for " sums products of q -Euler numbers and polynomials" by using fermionic p -adic q -integral on \mathbb{Z}_p .

2. q -Extension of Euler numbers

In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. Now we consider the q -Extension of Euler polynomials as follows:

$$F_q(x, t) = \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} \frac{E_{n,q}(x)}{n!} t^n, \quad |t + \log q| < \pi. \quad (2.1)$$

Note that

$$\lim_{q \rightarrow 1} F_q(x, t) = F(x, t) = \frac{2}{e^t + 1} e^{xt} = \sum_{n=0}^{\infty} \frac{E_n(x)}{n!} t^n.$$

In the special case $x = 0$ the q -Euler polynomial $E_{n,q}(0) = E_{n,q}$ will be called q -Euler numbers. It is easy to see that $F_q(x, t)$ is analytic function in \mathbb{C} . Hence we have

$$\sum_{n=0}^{\infty} \frac{E_{n,q}(x)}{n!} t^n = \frac{[2]_q}{qe^t + 1} e^{xt} = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{(n+x)t}. \quad (2.2)$$

If we take the k -th derivative at $t = 0$ on bosides in (2.2), then we have

$$E_{k,q}(x) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n (n+x)^k. \quad (2.3)$$

From (2.3) we can define q -zeta function which interpolating q -Euler numbers at negative integer as follows:

For $s \in \mathbb{C}$, we define

$$\zeta_q(s, x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^n}{(n+x)^s}, \quad s \in \mathbb{C}. \quad (2.4)$$

Note that $\zeta_q(s, x)$ is analytic in complex s -plane. If we take $s = -k$ ($k \in \mathbb{Z}_+$), then we have $\zeta_q(-k, x) = E_{k,q}(x)$.

By (2.3) and (2.4), we obtain the following:

Theorem 1. For $k \in \mathbb{Z}_+$, we have

$$E_{k,q}(x) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n (n+x)^k.$$

Let $F_q(0, t) = F_q(t)$. Then we see that

$$\begin{aligned} [2]_q \sum_{k=0}^{n-1} (-1)^k q^k e^{kt} &= \frac{[2]_q}{1 + qe^t} - [2]_q \frac{(-1)^n q^n e^{nt}}{1 + qe^t} \\ &= F_q(t) - (-1)^n q^n F_q(n, t). \end{aligned} \quad (2.5)$$

From (2.5), we derive

$$\sum_{k=0}^{\infty} \left([2]_q \sum_{l=0}^{n-1} (-1)^l q^l l^k \right) \frac{t^k}{k!} = \sum_{k=0}^{\infty} (E_{k,q} - (-1)^n q^n E_{k,q}(n)) \frac{t^k}{k!}. \quad (2.6)$$

By comparing the coefficients on both sides in (2.6), we obtain the following:

Theorem 2. Let $n \in \mathbb{N}, k \in \mathbb{Z}_+$. If $n \equiv 0 \pmod{2}$, then we have

$$E_{k,q} - q^n E_{k,q}(n) = [2]_q \sum_{k=0}^{n-1} (-1)^l q^l l^k.$$

If $n \equiv 1 \pmod{2}$, then we have

$$E_{k,q} + q^n E_{k,q}(n) = [2]_q \sum_{k=0}^{n-1} (-1)^l q^l l^k.$$

For $w_1, w_2, \dots, w_r \in \mathbb{C}$, we consider the multiple q -Euler polynomials of Barnes type as follows:

$$\begin{aligned} F_q^r(w_1, w_2, \dots, w_r | x, t) &= \frac{[2]_q^r e^{xt}}{(qe^{w_1 t} + 1)(qe^{w_2 t} + 1) \cdots (qe^{w_r t} + 1)} \\ &= \sum_{n=0}^{\infty} E_{n,q}(w_1, \dots, w_r | x) \frac{t^n}{n!}, \text{ where } \max_{1 \leq i \leq r} |w_i t + \log q| < \pi \end{aligned} \quad (2.7)$$

For $x = 0$, $E_{n,q}(w_1, \dots, w_r | 0) = E_{n,q}(w_1, \dots, w_r)$ will be called the multiple q -Euler numbers of Barnes type. It is easy to see that $F_q^r(w_1, w_2, \dots, w_r | x, t)$ is analytic function in the given region. From (2.7) we derive

$$[2]_q^r \sum_{n_1, \dots, n_r=0}^{\infty} (-q)^{\sum_{i=1}^r n_i} e^{(\sum_{i=1}^r n_i w_i + x)t} = \sum_{n=0}^{\infty} E_{n,q}(w_1, \dots, w_r | x) \frac{t^n}{n!}. \quad (2.8)$$

By the k -th differentiation on both sides in (2.8), we see that

$$[2]_q^r \sum_{n_1, \dots, n_r=0}^{\infty} (-q)^{\sum_{i=1}^r n_i} \left(\sum_{i=1}^r n_i w_i + x \right)^k = E_{k,q}(w_1, \dots, w_r | x). \quad (2.9)$$

From (2.8), we can derive the following Barnes' type multiple q -Euler zeta function as follows:

For $s \in \mathbb{C}$, define

$$\zeta_{r,q}(w_1, w_2, \dots, w_r | s, x) = \sum_{n_1, \dots, n_r=0}^{\infty} \frac{(-1)^{n_1 + \dots + n_r} q^{n_1 + \dots + n_r}}{(n_1 w_1 + n_2 w_2 + \dots + n_r w_r + x)^s}. \quad (2.10)$$

By (2.9) and (2.10), we obtain the following:

Theorem 3. For $k \in \mathbb{Z}_+, w_1, w_2, \dots, w_r \in \mathbb{C}$, we have

$$\zeta_{r,q}(w_1, w_2, \dots, w_r | -k, x) = E_{k,q}(w_1, w_2, \dots, w_r | x).$$

Let χ be the primitive Dirichlet character with conduct $f(=\text{odd}) \in \mathbb{N}$. Then we consider a generalized Euler numbers attached to χ as follows:

$$F_{\chi,q}(t) = \frac{[2]_q \sum_{a=0}^{f-1} (-1)^a q^a \chi(a) e^{at}}{q^f e^{ft} + 1} = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!}, \quad (2.11)$$

where $|\log q + t| < \frac{\pi}{f}$. The numbers $E_{n,\chi,q}$ will be called the generalized q -Euler numbers attached to χ . From (2.11), we note that

$$\begin{aligned} F_{\chi,q}(t) &= \frac{[2]_q \sum_{a=0}^{f-1} (-1)^a q^a \chi(a) e^{at}}{q^f e^{ft} + 1} \\ &= [2]_q \sum_{a=0}^{f-1} (-1)^a q^a \chi(a) \sum_{n=0}^{\infty} q^{nf} (-1)^n e^{(a+nf)t} \\ &= [2]_q \sum_{n=0}^{\infty} \sum_{a=0}^{f-1} (-1)^{a+nf} q^{a+nf} \chi(a+nf) e^{(a+nf)t} \\ &= [2]_q \sum_{n=0}^{\infty} (-1)^n q^n \chi(n) e^{nt} = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!}. \end{aligned}$$

Thus, we have

$$E_{n,\chi,q} = \left. \frac{d^k}{dt^k} F_{\chi,q}(t) \right|_{t=0} = [2]_q \sum_{n=1}^{\infty} (-1)^n q^n \chi(n) n^k, \quad (k \in \mathbb{N}).$$

Therefore, we can define the Dirichlet's type l -function which interpolates at negative integer as follows:

For $s \in \mathbb{C}$, we define $l_q(s, \chi)$ as

$$l_q(s, \chi) = [2]_q \sum_{n=1}^{\infty} \frac{(-1)^n q^n \chi(n)}{n^s}. \quad (2.13)$$

By (2.12) and (2.13), we obtain the following:

Theorem 4. For $k \in \mathbb{Z}_+$, we have

$$l_q(-k, \chi) = E_{k,\chi,q}.$$

From (2.1) and the definition of q -Euler numbers, we derive

$$\begin{aligned} F_q(t, x) &= \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!} \sum_{l=0}^{\infty} \frac{x^l}{l!} t^l \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^m E_{n,q} \binom{m}{n} x^{n-l} \right) \frac{t^m}{m!}. \end{aligned} \quad (2.14)$$

By (2.14) we see that

$$E_{n,q}(x) = \sum_{m=0}^n E_{m,q} \binom{n}{m} x^{n-m}, \quad n \in \mathbb{Z}_+. \quad (2.15)$$

For f ($=\text{odd}$) $\in \mathbb{N}$, we note that

$$\begin{aligned} \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!} &= \frac{[2]_q}{qe^t + 1} e^{xt} \\ &= [2]_q \frac{1}{q^f e^{ft} + 1} \sum_{a=0}^{f-1} (-1)^a q^a e^{\left(\frac{a+x}{f}\right)ft} \\ &= \frac{[2]_q}{[2]_{q^f}} \sum_{a=0}^{f-1} (-1)^a q^a \left(\frac{[2]_{q^f} e^{\left(\frac{a+x}{f}\right)ft}}{q^f e^{ft} + 1} \right) \\ &= \frac{[2]_q}{[2]_{q^f}} \sum_{a=0}^{f-1} (-1)^a q^a \sum_{n=0}^{\infty} E_{n,q^f} \left(\frac{a+x}{f} \right) \frac{f^n t^n}{n!}. \end{aligned}$$

Thus, we have the distribution relation for q -Euler polynomials as follows:

Theorem 5. For f ($=\text{odd}$) $\in \mathbb{N}$, we have

$$E_{n,q}(x) = \frac{f^n [2]_q}{[2]_{q^f}} \sum_{a=0}^{f-1} (-1)^a q^a E_{n,q^f} \left(\frac{a+x}{f} \right).$$

For $k, n \in \mathbb{N}$ with $n \equiv 0 \pmod{2}$, it is easy to see that

$$\begin{aligned} [2]_q \sum_{l=0}^{n-1} (-1)^{l-1} q^l l^k &= q^n E_{k,q}(n) - E_{k,q} \\ &= q^n \sum_{m=0}^k \binom{k}{m} n^{k-m} E_{m,q} - E_{k,q} = q^n \sum_{m=0}^{k-1} \binom{k}{m} E_{m,q} n^{k-m} + (q^n - 1) E_{k,q}. \end{aligned}$$

Therefore, we obtain the following

Theorem 6. For $k, n \in \mathbb{N}$ with $n \equiv 0 \pmod{2}$, we have

$$[2]_q \sum_{l=0}^{n-1} (-1)^{l-1} q^l l^k = q^n \sum_{m=0}^{k-1} \binom{k}{m} E_{m,q} n^{k-m} + (q^n - 1) E_{k,q}.$$

3. Witt's type formulae on \mathbb{Z}_p in p -adic number field

In this section we assume that $q \in \mathbb{C}_p$ with $|1 - q|_p < 1$. g is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $g \in UD(\mathbb{Z}_p)$, if the difference quotient

$$F_g(x, y) = \frac{g(x) - g(y)}{x - y},$$

has a limit $f'(a)$ as $(x, y) \rightarrow (a, a)$. For $g \in UD(\mathbb{Z}_p)$, an invariant p -adic q -integral is defined as

$$I_q(g) = \int_{\mathbb{Z}_p} g(x) d\mu_q(x) = \lim_{N \rightarrow \infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} g(x) q^x.$$

The fermionic p -adic q -integral is also defined as

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N \rightarrow \infty} \frac{[2]_q}{1 + q^{p^N}} \sum_{x=0}^{p^N-1} g(x) (-1)^x q^x, \text{ see [4]}. \quad (3.1)$$

From (3.1), we have the integral equation as follows:

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q f(0), \text{ where } g_1(x) = g(x+1). \quad (3.2)$$

If we take $f(x) = e^{tx}$, then we have

$$I_q(e^{tx}) = \int_{\mathbb{Z}_p} e^{xt} d\mu_{-q}(x) = \frac{[2]_q}{qe^t + 1}. \quad (3.3)$$

From (3.3), we note that

$$\sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) \frac{t^n}{n!} = \frac{[2]_q}{qe^t + 1} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{n!}.$$

By comparing the coefficient on the both sides, we see that

$$\int_{\mathbb{Z}_p} x^n d\mu_{-q}(x) = E_{n,q}, n \in \mathbb{Z}_+. \quad (3.4)$$

By the same method we see that

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-q}(x) = \frac{[2]_q}{qe^t + 1} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}.$$

Hence, we have the formula of Witt's type for q -Euler polynomial as follows:

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-q}(x) = E_{n,q}, n \in \mathbb{Z}_+.$$

For $n \in \mathbb{Z}_+$, let $g_n(x) = g(x+n)$. Then we have

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l). \quad (3.5)$$

If n is odd positive integer, then we have

$$q^n I_{-q}(g_n) + I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^l q^l g(l). \quad (3.6)$$

Let χ be the primitive Dirichlet character with conduct $f(=\text{odd}) \in \mathbb{N}$ and let $g(x) = \chi(x)e^{xt}$. From (3.3) we derive

$$I_{-q}(\chi(x)e^{xt}) = \frac{[2]_q \sum_{a=0}^{f-1} (-1)^a q^a \chi(a) e^{at}}{q^f e^{ft} + 1} = \sum_{n=0}^{\infty} E_{n,\chi,q} \frac{t^n}{n!}.$$

Thus, we have the Witt's formula for generalized q -Euler numbers attached to χ as follows:

$$\int_{\mathbb{Z}_p} \chi(x) x^n d\mu_{-q}(x) = E_{n,\chi,q}, n \geq 0.$$

4. Higher order q -Euler numbers and polynomials

In this section we also assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. Now we study on higher order q -Euler numbers and polynomials and sums of products of q -Euler numbers. First, we try to consider the multivariate fermionic p -adic q -integral on \mathbb{Z}_p as follows:

$$\begin{aligned} & \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} e^{(a_1 x_1 + a_2 x_2 + \cdots + a_r x_r)t} e^{xt} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_r) \\ &= \frac{[2]_q^r}{(qe^{a_1 t} + 1)(qe^{a_2 t} + 1) \cdots (qe^{a_r t} + 1)} e^{xt}, \end{aligned} \quad (4.1)$$

where $a_1, a_2, \dots, a_r \in \mathbb{Z}_p$.

From (4.1) we consider the multiple q -Euler polynomials as follows:

$$\frac{[2]_q^r}{(qe^{a_1 t} + 1)(qe^{a_2 t} + 1) \cdots (qe^{a_r t} + 1)} e^{xt} = \sum_{n=0}^{\infty} E_{n,q}(a_1, a_2, \dots, a_r | x) \frac{t^n}{n!}. \quad (4.2)$$

In the special case $(a_1, a_2, \dots, a_r) = (1, 1, \dots, 1)$, we write

$$E_{n,q}(\underbrace{a_1 \cdots a_r}_{r \text{ times}} | x) = E_{n,q}^{(r)}(x).$$

For $x = 0$, the multiple q -Euler polynomials will be called as q -Euler numbers of order r .

For (4.2) and (4.2) we note that

$$E_{n,q}(a_1, a_2, \dots, a_r | x) = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{r \text{ times}} (a_1 x_1 + \cdots + a_r x_r + x)^n \prod_{j=1}^r d\mu_{-q}(x_j).$$

It is easy to check that

$$E_{n,q}(a_1, a_2, \dots, a_r | x) = \sum_{l=0}^n \binom{n}{l} x^{n-l} E_{l,q}(a_1, a_2, \dots, a_r),$$

where $E_{n,q}(a_1, a_2, \dots, a_r) = E_{n,q}(a_1, a_2, \dots, a_r | 0)$. Multinomial theorem is well known as follows:

$$\left(\sum_{j=1}^r x_j \right)^n = \sum_{\substack{l_1, \dots, l_r \geq 0 \\ l_1 + \dots + l_r = n}} \binom{n}{l_1, \dots, l_r} \prod_{a=1}^r x_a^{l_a},$$

where

$$\binom{n}{l_1, \dots, l_r} = \frac{n!}{l_1! l_2! \cdots l_r!}.$$

By (4.2) and (4.3) we easily see that

$$E_{n,q}^{(r)}(x) = \sum_{m=0}^n \sum_{\substack{l_1, \dots, l_r \geq 0 \\ l_1 + \dots + l_r = m}} \binom{n}{m} \binom{m}{l_1, \dots, l_r} x^{n-m} \prod_{j=1}^r E_{l_j, q}.$$

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